CONFORMAL FIELD THEORY AND GRAPHS *

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The ADE graphs – the root diagrams of the simply laced simple Lie algebras appeared in the framework of the two-dimensional conformal field theories (CFT) based on the affine algebra $\widehat{sl}(2)_k$ in the classification of modular invariant partition functions [1] and in some related lattice models [2]. The modular invariants are labelled by the Coxeter number $h = k + 2$ while their diagonal terms, encoding the scalar field content of the theory, are enumerated by the Coxeter exponents. Graphs describing the spectrum of some invariants for the $\widehat{sl}(n)$ theories were proposed in [3] ($\mathbb{Z}_n-$ orbifolds) and in [4] (some cases of $\widehat{sl}(3)$). The aim of the reviewed work [5] is to further extend these results exploiting some deeper relations with structures in CFT.

We first recall some basic facts. The WZNW genus one partition functions are sesquilinear forms on the affine algebra characters with integer coefficients $\mathcal{N}_{\lambda\,\bar\lambda}$,

$$Z^{(n+k)}(q) = \sum_{\lambda,\bar\lambda} \mathcal{N}_{\lambda\,\bar\lambda}\,\chi_{\lambda}(q)\,\chi_{\bar\lambda}(\bar q)\,, \qquad \lambda\,,\,\bar\lambda \in \mathcal{P}_{++}^{(k+n)}\,, \quad \mathcal{N}_{\rho\,\rho} = 1\,. \tag{1}$$

Here $\mathcal{P}_{++}^{(k+n)}$ is set of integrable heighest weights of the affine algebra $\widehat{sl}(n)_k$ (shifted by $\rho = (1, 1, \ldots, 1)$). The Type I theories are those for which there exists an algebra extending the chiral algebra and the partition function may be recast in a block-diagonal form

$$Z^{(n+k)} = \sum_i |\sum_{\lambda\in\mathcal{P}_{++}^{(k+n)}} \mathrm{mult}_{\mathcal{B}_i}(\lambda)\,\chi_{\lambda}|^2 = \sum_{\mathcal{B}_i} |\chi_{\mathcal{B}_i}|^2\,, \tag{2}$$

where $\mathcal{B}_i$ ($\chi_{\mathcal{B}_i}$) are representations (characters) of the extended chiral algebra and $\mathrm{mult}_{\mathcal{B}_i}(\lambda)$ is the multiplicity of $\lambda \in \mathcal{P}_{++}^{(k+n)}$ in $\mathcal{B}_i$. The modular invariance of (2) implies the relation

$$\sum_{\omega\in\mathcal{P}_{++}^{(h)}} \mathrm{mult}_{\mathcal{B}_i}(\omega)\,S_{\gamma\omega} = \sum_{\mathcal{B}_j} \mathrm{mult}_{\mathcal{B}_j}(\gamma)\,S_{\mathcal{B}_j\,\mathcal{B}_i}\,, \tag{3}$$

where $S_{\gamma\omega}$ and $S_{\mathcal{B}_j\,\mathcal{B}_i}$ are the initial and extended modular matrices; $\mathrm{mult}_{\mathcal{B}_i}(\rho) = \delta_{\mathcal{B}_i\mathbf{1}}$.

We postulate that each of the looked for graphs $\mathcal{G}$ satisfies a set of requirements: namely it is connected and unoriented, i.e., described by an irreducible, symmetric adjacency matrix ${}^tG = G$ with G_{ab}–nonnegative integers; in the set of vertices $\mathcal{V}$ a $\mathbb{Z}/n\mathbb{Z}$ grading $a \mapsto \tau(a)$, the "n-ality", is introduced, so that $G_{ab} \neq 0$ only if $\tau(a) \neq \tau(b)$. This enables

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one to split this adjacency matrix into a sum of $n-1$ matrices $G = G_1 + G_2 + \cdots + G_{n-1}$, where $(G_p)_{ab} \neq 0 \quad$ only if $\quad \tau(b) = \tau(a) + p \bmod n$, and ${}^tG_p = G_{n-p}$. Accordingly, the graph may be regarded as the superposition on the same set of vertices of $n-1$ oriented (except for $p = n/2$), not all necessarily connected, graphs $\mathcal{G}_p$ of adjacency matrices G_p, $p = 1, \cdots, n-1$. We will furthermore require that there exists an involution $a \mapsto a^\vee$ on $\mathcal{V}$ such that $\tau(a^\vee) = -\tau(a)$ and $(G_p)_{ab} = (G_p)_{b^\vee a^\vee}$. The matrices G_p are assumed to commute among themselves, hence are "normal", i.e. simultaneously diagonalisable in an orthonormal basis; the common eigenvectors are labelled by integrable weights $\lambda \in \mathcal{P}_{++}^{(k+n)}$ for some level k, we denote them $\psi^\lambda = (\psi_a^{(\lambda)})$, $a \in \mathcal{V}$. The set of these λ, some of which may occur with multiplicities larger than one, will be denoted by Exp. We require that the eigenvalues $\gamma_p^{(\lambda)}$ of G_p coincide with the corresponding eigenvalues of the Verlinde fundamental matrices $N_{\hat{\Lambda}_p+\rho}$, i.e., $\gamma_p^{(\lambda)} = S_{\hat{\Lambda}_p+\rho\,\lambda}/S_{\rho\,\lambda}$, where $\hat{\Lambda}_1, \cdots, \hat{\Lambda}_{n-1}$ are the fundamental weights. It is assumed that the identity representation $\rho \in \mathrm{Exp}$ and that it appears with multiplicity one: it corresponds to the eigenvector of largest eigenvalue of G_1, $\gamma_1^{(\rho)} \geq |\gamma_1^{(\lambda)}|$, the so–called Perron–Frobenius eigenvector; its components $\psi_a^{(\rho)}$, $a \in \mathcal{V}$, are positive. The eigenvector matrices $\psi_a^{(\lambda)}$ replace the (symmetric) modular matrices $S_{\mu\lambda}$ of the diagonal theory and satisfy the relations

$$\sum_{a\in\mathcal{V}} \psi_a^{(\lambda)}\, \psi_a^{(\nu)\,*} = \delta_{\lambda\nu}\,, \qquad \sum_{\lambda\in\mathrm{Exp}} \psi_a^{(\lambda)}\, \psi_c^{(\lambda)\,*} = \delta_{ac}\,, \qquad \psi_{a^\vee}^{(\lambda)} = \psi_a^{(\lambda)\,*} = \psi_a^{(\lambda^*)}\,.$$

It is assumed that there is at least one vertex denoted by $\mathbf{1}$, $\mathbf{1}^\vee = \mathbf{1}$, $\tau(\mathbf{1}) = 0$, such that $\psi_{\mathbf{1}}^\lambda > 0$ for all $\lambda \in \mathrm{Exp}$.

Accordingly one introduces two sets of real numbers providing two extensions of the Verlinde formula for the fusion rule multiplicities,

$$N_{ab}^{\ c} = \sum_{\lambda\in\mathrm{Exp}} \frac{\psi_a^{(\lambda)}\psi_b^{(\lambda)}\psi_c^{(\lambda)\,*}}{\psi_{\mathbf{1}}^{(\lambda)}}\,, \qquad M_{\lambda\mu}^{\ \ \nu} = \sum_{a\in\mathcal{V}} \frac{\psi_a^{(\lambda)}\,\psi_a^{(\mu)}\,\psi_a^{(\nu)\,*}}{\psi_a^{(\rho)}}\,. \tag{4}$$

These two sets of real numbers (assumed furthermore to be nonnegative for the Type I theories) can be looked as the structure constants of a pair of associative, commutative algebras $\mathcal{U}$ and $\hat{\mathcal{U}}$, with identity and involution, and with $|\mathcal{V}| = |\mathrm{Exp}|$ one-dimensional representations provided by the eigenvalues, i.e., as a dual pair of "$C-$algebras", [6]. In particular in the diagonal cases the two algebras are selfdual and coincide, the diagonalisation matrix $\psi_a^{(\lambda)}$ being replaced by the symmetric matrix $S_{\mu\lambda}$. The relevance of the $C-$algebras with nonnegative structure constants for the study of Type I theories was first pointed out and exploited in [4]. There is another generalisation of the Verlinde formula, also studied extensively in [4], namely the set of integers defined according to

$$V_{\gamma a}^{b} = \sum_{\omega\in\mathrm{Exp}} \frac{S_{\gamma\omega}}{S_{\rho\omega}}\, \psi_a^{(\omega)}\, \psi_b^{(\omega)\,*}\,, \qquad a, b \in \mathcal{V}\,, \quad \gamma \in \mathcal{P}_{++}^{(k+n)}\,, \tag{5}$$

in particular $V_{\hat{\Lambda}_p+\rho} = G_p$. The matrices V_λ realise a representation of the Verlinde fusion algebra, while V_b for any $b \in \mathcal{V}$, considered as a rectangular matrix $(V_b)_\lambda^c = V_{\lambda b}^c$, intertwines

the adjacency matrices of the diagonal and the nondiagonal graphs for a given value of the level k, i.e., $N_{\hat{\Lambda}_p+\rho}\, V_b = V_b\, G_p$. Note also the relation

$$V_\gamma\, N_a = \sum_{b\in\mathcal{V}} V_{\gamma a}^{b}\, N_b\,, \quad \text{or}\,, \quad V_\gamma = \sum_{b\in\mathcal{V}} V_{\gamma\mathbf{1}}^{b}\, N_b\,, \tag{6}$$

which implies that the adjacency matrices G_p belong to the N algebra being expressed as linear combinations (with nonnegative integer coefficients) of the basis $\{N_a\}$.

Along with a few exceptional cases there are (restricting to embeddings into simple algebras $\hat{g}_1$) four infinite series of conformal embeddings of the algebra $\widehat{sl}(n)$ [7]

$$\begin{gathered}
\widehat{sl}(n)_{n-2} \subset \widehat{sl}\Big(\frac{n(n-1)}{2}\Big)_1\,, \qquad n\geq 4\,,\\
\widehat{sl}(n)_{n+2} \subset \widehat{sl}\Big(\frac{n(n+1)}{2}\Big)_1\,,\\
\widehat{sl}(2n+1)_{2n+1} \subset \widehat{so}(4n(n+1))_1\,,\\
\widehat{sl}(2n)_{2n} \subset \widehat{so}(4n^2-1)_1\,, \qquad n\geq 2\,.
\end{gathered}\tag{7}$$

The set Exp splits into classes B_i – the integrable representations of $\hat{g}_1$. We shall also denote the elements in Exp by $(\lambda, i, \varepsilon_i)$, $\lambda\in\mathcal{B}_i$, $|\varepsilon_i| = \mathrm{mult}_{\mathcal{B}_i}(\lambda)$, to distinguish $\lambda\in\mathcal{P}_{++}^{(k+n)}$ appearing in different extended algebra representations as well as with a nontrivial multiplicity within a given extended representation; for simplicity of notation the indices i,ε are sometimes omitted. The physical fields are labelled by pairs $\Big((\lambda,i,\varepsilon_i)\,,\,(\bar\lambda,i,\bar\varepsilon_i)\Big)$, $\lambda,\bar\lambda\in\mathcal{B}_i$.

It was observed in [5] that in the case $sl(2)$ the nondiagonal solutions for the relative structure constants accounting for the contribution of the scalar fields in the expansion of products of scalar physical operators coincide with the set of algebraic numbers $M_{\lambda\mu}^{\gamma}$ in (4) for the corresponding $D-$, or $E-$ type graph. Furthermore in the cases described by conformal embeddings the same numbers determine uniquely the more general spin fields structure constants, since the (square of the) latter factorise into a left and right chiral parts precisely given by a pair of $M-$ algebra structure constants. This property of the models related to conformal embeddings allows to block-diagonalise the duality equations for the relative structure constants thus recovering the extended model fusion (crossing) matrices appearing in the corresponding diagonal equations. In a more restricted version this idea has been worked out and extended for the higher rank cases $\widehat{sl}(n)$ in [5] assuming that in all theories described by conformal embeddings a kind of chiral factorisation of the general structure constants takes place. This led us to the following set of equations for the chiral pieces denoted $M_{\lambda\mu}^{\gamma}$ by analogy with the $n=2$ case

$$N_{\mathcal{B}_i\,\mathcal{B}_k}^{\mathcal{B}_j} = \sqrt{\frac{D_{\mathcal{B}_i}}{D_\lambda}}\sqrt{\frac{D_{\mathcal{B}_k}}{D_\mu}}\sum_{\gamma\in\mathcal{P}_{++}^{(k+n)}\,;\,\varepsilon_j} M_{(\lambda,i,\varepsilon_i)\,(\mu,k,\varepsilon_k)}^{(\gamma,j,\varepsilon_j)}\sqrt{\frac{D_\gamma}{D_{\mathcal{B}_j}}}\,, \qquad \forall\ \lambda\in\mathcal{B}_i\,,\ \mu\in\mathcal{B}_k\,. \tag{8}$$

Here in the l.h.s. $N_{\mathcal{B}_i\,\mathcal{B}_k}^{\mathcal{B}_j}$ is the extended Verlinde fusion rule multiplicity, while the quantum dimensions D_λ and $D_{\mathcal{B}_i}$ of the initial and the extended theories appear in the r.h.s.; this data is known for all of the conformal embeddings–in particular the extended multiplicity in the l.h.s. of (8) takes only the values $0,1$. The constants $M_{(\lambda,i,\varepsilon_i)\,(\mu,k,\varepsilon_k)}^{(\gamma,j,\varepsilon_j)} \equiv 0$ if the Verlinde multiplicies $N_{\lambda\,\mu}^{\gamma}$, or $N_{\mathcal{B}_i\,\mathcal{B}_k}^{\mathcal{B}_j}$ vanish; the summation in the r.h.s. of (8) is restricted within the class $\mathcal{B}_j$. In the case $n=3$ the solutions of the set of algebraic equations (8) are consistent with the values for the $M-$ algebra matrix elements (4) found in [4] and furthermore they lead to new solutions for higher n, allowing to construct the corresponding graphs, see [5] for explicit examples.

The set of equations (8) furthermore allows to determine explicitly in general some of the eigenvalues of the $M-$ matrices, i.e., some of the $1-$ dimensional representations of the $M-$ algebra. To do that one establishes a one to one correspondence between the set of integrable representations of the extended Kac–Moody algebra $\hat{g}_1$ and a subset $T \subset \mathcal{V}$ of the vertices of the graph such that if $c \in T$ is identified with a representation $\mathcal{B}_i$, then $c^\vee$ is also in T and is identified with $\mathcal{B}_i^*$; we shall identify the vertex $\mathbf{1}$ with the identity weight of the extended algebra. One obtains then from (8) an analytic formula for $\psi_c^{(\lambda,i,\varepsilon)}/\psi_c^{(\rho)}$, for $\forall\, c \in T$, $(\lambda,i,\varepsilon_i) \in \mathrm{Exp}$, in terms of the extended modular matrices $S_{c\ B_i} \equiv S_{B_j B_i}$ $(c \equiv B_j)$, which furthermore implies

$$\frac{\psi_c^{(\lambda,i,\varepsilon)}}{\psi_{\mathbf{1}}^{(\lambda,i,\varepsilon)}} = \frac{S_{c\,\mathcal{B}_i}}{S_{\mathbf{1}\,\mathcal{B}_i}}\,, \qquad \psi_{\mathbf{1}}^{(\lambda,i,\varepsilon)} = \sqrt{S_{\rho\,\lambda}\, S_{\mathbf{1}\,\mathcal{B}_i}}\,, \qquad c \in T\,, (\lambda,i,\varepsilon_i) \in \mathrm{Exp}\,. \tag{9}$$

According to (9) the (dual) Perron-Frobenius eigenvector $\psi_{\mathbf{1}}$ has indeed positive components. The formula (9) provides general explicit expression for the subset of components of the eigenvectors of the adjacency matrices corresponding to the subset T. This allows to determine the matrix elements N_{ab}^c for $a,b,c \in T$ as coinciding with the corresponding extended Verlinde matrix elements $N_{\mathcal{B}_i\,\mathcal{B}_k}^{\mathcal{B}_j}$ for $a \equiv \mathcal{B}_i\,, b \equiv \mathcal{B}_k\,, c \equiv \mathcal{B}_j$. Furthermore assuming that both N and M structure constants are nonnegative one shows that the set $N_a\,, a \in T$, provides a subalgebra $\mathcal{U}_T$ of the $N-$ algebra $\mathcal{U}$ isomorphic to the extended Verlinde algebra $N_{\mathcal{B}_i}$. The set of vertices $\mathcal{V}$ then splits into equivalence classes $T_1 = T, T_2, ..., T_t$ which allows to define a $C-$ factor algebra $\mathcal{U}/\mathcal{U}_T$ [6]. Alternatively the $M-$ algebra $\hat{\mathcal{U}}$ has a $C-$ subalgebra $\hat{\mathcal{U}}_{\hat{T}}$, described by the subset $\hat{T}$ of Exp appearing in the decomposition of the identity representation of the extended algebra, while the factor $C-$ algebra $\hat{\mathcal{U}}/\hat{\mathcal{U}}_{\hat{T}}$ is isomorphic to the extended Verlinde algebra. The relation (8) then can be interpreted as expressing the structure constants of $\hat{\mathcal{U}}/\hat{\mathcal{U}}_{\hat{T}}$ as an average of the structure constants of $\hat{\mathcal{U}}$ over classes $\hat{T}_i$, i.e., a relation in the theory of $C-$ algebras with nonnegative structure constants [6], realised here with explicit specific values of the parameters, given by the quantum dimensions. Note that, although obtained under assumptions which sound plausible only for the cases of conformal embeddings, the relation (8) and its consequences presumably hold true also for the orbifold theories (where one can determine explicitly the full eigenvector matrices); we have checked this for the $\mathbb{Z}_n-$ orbifold graphs of [3], see also the recent work [8] for general expressions for the orbifolds extended modular matrices.

One further consequence of (8) is obtained inserting (9) into (5), namely

$$V_{\gamma\mathbf{1}}^{c} = \sum_{\mathcal{B}_i} S^{*}_{\mathcal{B}_j\mathcal{B}_i} \sum_{\omega\in\mathcal{P}_{++}^{(k+n)}} \mathrm{mult}_{\mathcal{B}_i}(\omega)\, S_{\gamma\omega} = \mathrm{mult}_{\mathcal{B}_j}(\gamma)\,, \qquad \mathcal{B}_j \equiv c\in T\,, \tag{10}$$

using for the second equality in (10) the consistency condition (3). This determines the matrix $N_{\lambda\bar\lambda}$ in (1) in terms of the intertwiner $V_{\mathbf{1}}$,

$$\mathcal{N}_{\lambda\bar\lambda} = \sum_{c\in T} V_{\lambda\mathbf{1}}^{c}\, V_{\bar\lambda\mathbf{1}}^{c}\,, \tag{11}$$

a property first empirically observed in [4], and also derived recently by Ocneanu [9] in a different context as reflecting the counting of "essential paths" on the graph.

The set of vertices for which the components $\psi_a^{(\mu)}$ are explicitly determined according to (9) can be enlarged beyond the subset T. Indeed using (9) and (3) we have

$$\sum_{a\in\mathcal{V}} (V_{\lambda\mathbf{1}}^{a})^2 = \sum_{\alpha\in\mathbf{1}} N_{\lambda\lambda^*}^{\alpha}\,, \tag{12}$$

where $N_{\lambda\lambda^*}^{\alpha}$ are Verlinde multiplicities, and hence

$$\forall\lambda\in\mathcal{P}_{++}^{(k+n)}\,,\text{s.t.}\sum_{\alpha\in\mathbf{1}} N_{\lambda\lambda^*}^{\alpha} = 1\,,\quad \exists\; a_\lambda\in\mathcal{V}\,,\text{ s.t. } V_{\lambda\mathbf{1}}^{a} = \delta_{a a_\lambda}\,. \tag{13}$$

Hence according to (6) $V_\lambda \equiv N_{a_\lambda}$. This allows to determine $\psi_{a_\lambda}^{(\mu)}$ using (9) and the fact that V_λ admits eigenvalues identical to eigenvalues of the Verlinde matrix N_λ, c.f. (5). An example is provided by the fundamental weight $\lambda = \hat\Lambda_1 + \rho$ for which the above condition can be checked [10] to hold for all embeddings, so that $(G_1)_{\mathbf{1}a} = \delta_{a a_{\hat\Lambda_1+\rho}}$ which implies that the vertex $\mathbf{1}$ is "extremal", i.e., there is only one arrow leaving it (or entering it), the corresponding vertex being $a_{\hat\Lambda_1+\rho}$ (and $a_{\hat\Lambda_{n-1}+\rho}$). The latter property of $\mathbf{1}$, assumed in [5], was proved in the recent paper [10] starting from a more abstract setting. In fact this property of $\mathbf{1}$ extends to all vertices $a\in T$, such that $D_a = D_{\mathcal{B}_i} = 1$: i.e., $(G_1)_{ab} = \delta_{a\,b(a)}$ for such a and some $b = b(a)$, and hence in the first three series in (7) it extends to the full set T, thus allowing to recover the components $\psi_b^{(\mu)}$ corresponding to the class $T_{a_{\hat\Lambda_1+\rho}} \ni b$.

We expect that taking into account properties as in the above observations suggested by the results in [5] and [10], will make possible the construction of many new examples of graphs. Let us add one such new example: consider the embedding $\widehat{sl}(4)_6 \subset \widehat{sl}(10)_1$ (see [11] for the explicit expression for the modular invariant). Denote the exponents $(\lambda, i)\,, i = 0, 1, \ldots, 9\,, |\mathrm{Exp}| = 32$. The graphs $\mathcal{G}_p$ are recovered from the following eigenvector matrix (with $\psi_{a_j}^{(\lambda,i)}$, $a_j\in T$, determined from (9), $a_0\equiv\mathbf{1}\equiv\mathcal{B}_0$, and $S_{\mathcal{B}_j\,\mathcal{B}_k} = \frac{1}{\sqrt{10}}\exp\left(\frac{2\pi i}{10}\, jk\right)$)

$$\psi_{b_j}^{(\lambda,i)} := \gamma_1^{(\lambda)}\,\psi_{a_j}^{(\lambda,i)}\,,\quad \psi_{c_j}^{(\lambda,i)} := \gamma_2^{(\lambda)}\,\psi_{a_j}^{(\lambda,i)}\,,\quad j = 0,1,\ldots,9\,,\quad a_j\in T\,,$$

$$\psi_{d_0}^{(\lambda,i)} := \frac{S_{(2,2,1)\,\lambda}}{S_{\rho\lambda}}\,\psi_{\mathbf{1}}^{(\lambda,i)} - \gamma_1^{(\lambda)}\,\psi_{a_9}^{(\lambda,i)}\,,\quad \psi_{d_5}^{(\lambda,i)} := \frac{S_{(5,2,2)\,\lambda}}{S_{\rho\lambda}}\,\psi_{\mathbf{1}}^{(\lambda,i)} - \gamma_1^{(\lambda)}\,\psi_{a_4}^{(\lambda,i)}\,. \tag{14}$$

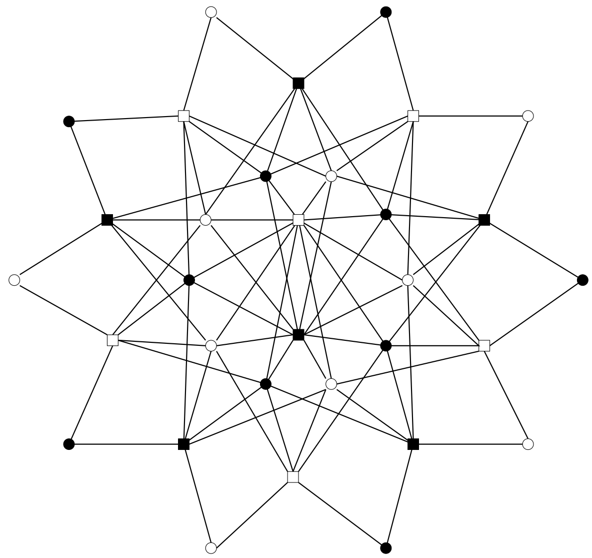

Fig. 1: The graph of G_1 corresponding to the conformal embedding $\widehat{sl}(4)_6 \subset \widehat{sl}(10)_1$. The points of 4-ality 0 and 2 (resp 1 and 3) are represented by open and black disks (resp squares). The involution $a \to a^\vee$ is the reflection in the horizontal diameter. The vertices denoted by $a_j\,, b_j\,, c_j\;\; (j = 0, 1, \ldots 9)\,,$ in (14) lie clockwise on three (inward going) concentric circles respectively, while d_0 and d_5 denote the selected (bottom and top respectively) vertices in the centre.

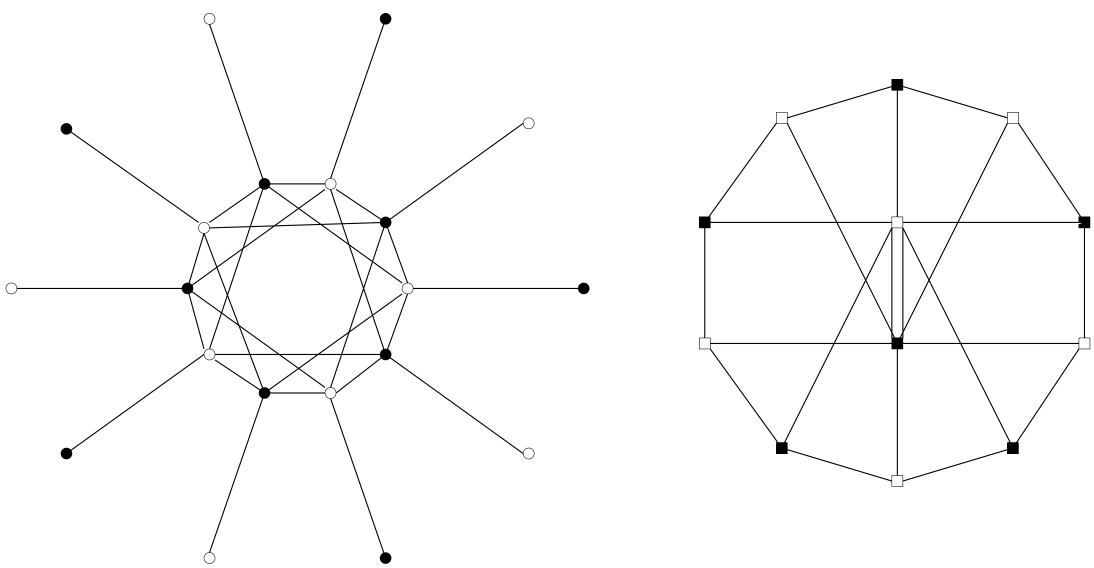

Fig. 2: Graph of G_2 for the same case.

We conclude with the remark that an example from the last series in (7) (see Appendix A of [5]) suggests that the above scheme might require some modifications – either extending the meaning of the graphs, allowing for noninteger values of the matrix elements of the

adjacency matrices, or, the meaning of the N - algebra, allowing for a noncommutative extension (see also [10]).

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